

# A Principle for Critical Point under Generalized Regular Constraint and Ill- Posed Lagrange Multipliers under Non-Regular Constraints

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**ABSTRACT.** In this paper, a kind of non regular constraints and a principle for seeking critical point under the constraint are presented, where no Lagrange multiplier is involved. Let  $E, F$  be two Banach spaces,  $g : E \rightarrow F$  a  $C^1$  map defined on an open set  $U$  in  $E$ , and the constraint  $S = \text{the preimage } g^{-1}(y_0), y_0 \in F$ . A main difference between the non regular constraint and regular constraint is that  $g'(x)$  at any  $x \in S$  is not surjective. Recently, the critical point theory under the non regular constraint is a concerned focus in optimization theory. The principle also suits the case of regular constraint. Coordinately, the generalized regular constraint is introduced, and the critical point principle on generalized regular constraint is established. Let  $f : U \rightarrow \mathbb{R}$  be a nonlinear functional. While the Lagrange multiplier  $L$  in classical critical point principle is considered, and its expression is given by using generalized inverse  $g'^+(x)$  of  $g'(x)$  as follows : if  $x \in S$  is a critical point of  $f|_S$ , then  $L = f'(x) \circ g'^+(x) \in F^*$ . Moreover, it is proved that if  $S$  is a regular constraint, then the Lagrange multiplier  $L$  is unique; otherwise,  $L$  is ill-posed. Hence, in case of the non regular constraint, it is very difficult to solve Euler equations, however, it is often the case in optimization theory. So the principle here seems to be new and applicable. By the way, the following theorem is proved; if  $A \in B(E, F)$  is double split, then the set of all generalized inverses of  $A$ ,  $GI(A)$  is smooth diffeomorphic to certain Banach space. This is a new and interesting result in generalized inverse analysis.

**Keywords:** Critical point theory, Optimization theory, Ill-posed problem, Generalized regular constraint

# 1 INTRODUCTION AND PRELIMINARY

Let  $E, F$  be two Banach spaces and  $g : E \rightarrow F$  be a  $c^1$  map defined on an open set  $U$  in Banach space  $E$ . Recall that a point  $x \in U$  is said to be a regular point (or submersion) of  $g$  if the Frechét differential  $g'(x)$  is surjective and  $N(g'(x))$  splits  $E$ , and  $y_0 \in F$  is a regular value of  $g$  provided the preimage  $g^{-1}(y_0)$  is empty or consists only of regular points. It is known that the fundamental theorem on critical point theory under regular constraint holds:

**Theorem** (Preimage) If  $y_0 \in F$  is a regular value of  $c^1$  map  $g : U \rightarrow F$ , then the preimage  $S = g^{-1}(y_0)$  is a  $c^1$  submanifold of  $U$  with the tangent space  $T_x S = N(g'(x))$  for any  $x \in S$ . Where  $N(,)$  denotes the null space of the operator in the parenthesis. (For the details see [AMR] and [Z].)

Recently, the concept of regular point has been extended to generalized regular point, i.e.,  $x_0 \in U$  is said to be a generalized regular point of  $g$  provided  $g'(x_0)$  is double split,  $F = R(g'(x_0)) \oplus N_+$ , and there exists a neighborhood  $U_0 \subset U$  of  $x_0$  such that  $R(g'(x)) \cap N_+ = \{0\}$  for any  $x \in U_0$ , where  $R(,)$  denotes the range of the operator in the parentheses. Obviously, regular point and immersion both are generalized regular points; when the rank of  $g'(x_0)$ ,  $\text{Rank}(g'(x_0)) < \infty$ ,  $x_0$  is generalized regular point if and only if  $x_0$  is subimmersion. It is natural to define the generalized value  $y_0 \in F$  of  $g$  as  $S = g^{-1}(y_0)$  is empty or consists only of generalized regular points. We also have the following fundamental theorem of critical point theory under generalized regular constraint:

**Theorem** (Generalized Preimage) If  $y_0 \in F$  is generalized regular value of  $c^1$  map  $g$ , then  $S = g^{-1}(y_0)$  is a  $c^1$  submanifold of  $U$  with the tangent space  $T_x S = N(g'(x))$  for any  $x \in S$ . (For the details see [M6] and [AMR].)

Fortunately, we have the following complete rank theorem in advanced calculus to answer the locally conjugate problem proposed by Berger, M. in [B],  
**Theorem** (Rank) Suppose that  $g : U \subset E \rightarrow F$  is a  $c^1$  map,  $g'(x_0)$  is double split, and  $g(x_0) = y_0, x_0 \in U$ . The following conclusion holds: there exist two neighborhoods  $U_0$  at  $x_0$ ,  $V_0$  at 0, two local diffeomorphisms  $\varphi : U_0 \rightarrow \varphi(U_0)$  and  $\psi : V_0 \rightarrow \psi(V_0)$ , such that

$$\varphi(x_0) = 0, \varphi'(x_0) = I, \psi(0) = y_0, \psi'(0) = I,$$

and

$$g(x) = (\psi \circ g'(x_0) \circ \phi)(x)$$

for all  $x \in U_0$  if and only if  $x_0$  is a generalized regular point of  $g$ . (For details see [M3], [M7], [M8], [B], [AMR] and [Z].)

(Note that the question on rank theorem in advanced calculus initially is to find a sufficient condition such that the conclusion of the rank theorem above holds.)

There are many equivalent conditions for generalized regular points, which are convenient for analysis calculus. (For the details see [M1], [M3], [M9] and [HM].)

By Theorem (Rank), the first main result in this paper, a principle for seeking critical point under generalized regular constraint is given, in which no Lagrange multiplier is involved. Also, we present some simple examples to illustrate its application and significance. Let  $B(E, F)$  be the set of all linear bounded operators from  $E$  into  $F$ ;  $A \in B(E, F)$  is called double split provided  $R(A)$  is closed and there exist closed subspaces  $R_+ \subset E$  and  $N_+ \subset F$  such that  $E = N(A) \oplus R_+$  and  $F = R(A) \oplus N_+$ , where  $N(A)$  is the null space of  $A$ . It is known well that there exists a generalized inverse  $A^+ \in B(F, E)$  for any double split operator  $A$ , such that  $AA^+A = A$  and  $A^+AA^+ = A^+$ . When  $E$  and  $F$  both are Hilbert spaces, a generalized inverse  $A^+$  is said to be  $M - P$  inverse of  $A$  provided  $AA^+$  and  $A^+A$  both are self adjoint. It is also known that the  $M - P$  inverse of  $A$  is unique, (see [N]). Let  $q'^+(x)$  be a generalized inverse of  $g'(x)$ . Then the second result is given as follows, suppose that  $S$  is generalized regular constraint, and  $f : U \rightarrow \mathbb{R}$  is a non linear functional; if  $x$  is critical point of  $f|_S$ , then

$$f'(x) - f'(x) \circ q'^+(x) \circ q'(x) = 0 \quad \text{and} \quad g(x) = y_0.$$

This shows that the Lagrange multiplier  $L$  is a bounded linear functional on  $F$  and  $= f'(x) \circ q'^+(x)$  at the critical point  $x$  of  $f|_S$ ; moreover, when  $S$  is a regular constraint, so that  $R(g'(x)) = F$  for any  $x \in S$ , the Lagrange multiplier  $L$  is unique since  $f'(x)e = f'(x) \circ g'^+(x) \circ g'(x)e \forall e \in E$  at the critical point  $x$  of  $f|_S$ . Specially, when  $E$  and  $F$  both are Hilbert spaces,  $g'^+(x)$  can be a  $M - P$  inverse of  $g'(x)$ , which is unique for any  $x \in S$ . Finally, let  $A \in B(E, F)$  be double split, and  $GI(A)$  the set of all generalized inverses of  $A$ . The following theorem for generalized inverse analysis is proved:  $GI(A)$  is smooth diffeomorphic to some Banach space, and then, using some results and presented method we can prove that the Lagrange multiplier  $L$  under non regular constraint is ill-posed. From this one can observe that it is very difficult to solve Euler equations in the case of non regular constraint. Hence besides the foundation for critical point under generalized regular constraint the principle presented here for seeking critical points is a new and available way since the Lagrange multiplier is no longer involved in the principle.

## 2 A Principle for Seeking Critical Point

Let  $g : U \rightarrow F$  be a  $c^1$  map, and  $y_0 \in F$  a generalized regular value of  $g$ . By Theorem (Generalized Preimage),  $S = g^{-1}(y_0)$  is a  $c^1$  submanifold of  $F$ . In what follows,  $S$  will be said to be the generalized regular constraint. Let  $f$  be a non linear functional on  $U$ . In this section, we discuss the critical point of  $f$  under generalized regular constraint, and give a principle for seeking the critical point, while no Lagrange multiple is involved.

**Theorem 2.1** *If  $x \in U$  is a critical point of  $f|_S$ , where  $S = g^{-1}(y_0)$ , and  $y_0 \in F$  is a generalized regular value of  $g$ , then*

$$N(f'(x)) \supset N(g'(x)) \quad \text{and} \quad g(x) = y_0. \quad (2.1)$$

*Proof.* Suppose that  $x_0 \in S$  is critical point of  $f|_S$ . Since  $y_0$  is a generalized regular value,  $x_0$  is a generalized regular point of  $g$ . By Theorem (Rank) there exist neighborhoods  $U_0$  at  $x_0$ ,  $V_0$  at  $0$ , local diffeomorphisms  $\varphi : U_0 \rightarrow \varphi(U_0)$  and  $\psi : V_0 \rightarrow \psi(V_0)$ , such that

$$\varphi(x_0) = 0, \quad \varphi'(x_0) = I, \quad \psi(0) = y_0, \quad \psi'(0) = I,$$

and

$$g(x) = (\psi \circ g'(x_0) \circ \varphi)(x) \quad \forall x \in U_0.$$

Since  $\varphi : U_0 \rightarrow \varphi(U_0)$  is a diffeomorphism, and  $\varphi(x_0) = 0$ , it is clear that there exists a positive number  $\varepsilon_0$  such that the following relation for arbitrary fixed  $h \in N(g'(x_0))$  holds :  $x(t) = \varphi^{-1}(th) \subset U_0$  for  $|t| < \varepsilon_0$ . Then it follows

$$g(x(t)) = (\psi \circ g'(x_0) \circ \varphi)(\varphi^{-1}(th)) = (\psi \circ g'(x_0))(th) = \psi(0) = y_0$$

(note  $\psi(0) = y_0$ ), this shows the curve  $x(t)$ ,  $-\varepsilon_0 < t < \varepsilon_0$  lies in  $S$ , and so,  $t = 0$  is critical point of the function  $f(x(t))$ . Therefore,

$$0 = \frac{df(x(t))}{dt} \Big|_{t=0} = f'(x_0)(\varphi^{-1})'(0)h = f'(x_0)h \quad \forall h \in N(g'(x_0)).$$

This proves  $N(f'(x_0)) \supset N(g'(x_0))$ . □

Specially, when  $E$  is a Hilbert space, we have

**Theorem 2.2** *Suppose that  $E$  is a Hilbert space,  $S$  is a generalized regular constraint and  $f$  is a  $c^1$  non linear functional defined on  $U$ . Let  $x \in S$  and  $f'(x) \neq 0$ . If  $x$  is a critical point of  $f|_S$ , then there exists non zero vector  $e_*(x) \in E$  such that*

$$N(g'(x)) \perp e_*(x) \quad \text{and} \quad N(f'(x)) \perp e_*(x).$$

*Proof.* Since  $f'(x) \neq 0$ , it is clear that  $Nf'(x) \not\subseteq E$ . Hence there exists a non zero vector  $e_*(x) \in E$  such that  $e_*(x) \perp Nf'(x)$ . By Theorem 2.1,  $e_*(x) \perp N(g'(x))$ .  $\square$

The following examples illustrate an application and significance of Theorem 2.1, although all of them are very simple.

**Example 1** Suppose that  $S$  is the unit circle in  $\mathbb{R}^2$ ,  $x^2 + y^2 = 1$ ,  $r_0 = \sqrt{x_0^2 + y_0^2} > 0$ , and  $f(x, y) = (x - x_0)^2 + (y - y_0)^2$ . By Theorem 2.1 go to find the extreme point  $(x, y)$  of  $f|_S$ .

It is clear that 1 is a regular value of  $g(x, y) = x^2 + y^2$ . By computing simple

$$N(f'(x, y)) = \{(\Delta x, \Delta y) : (x - x_0)\Delta x + (y - y_0)\Delta y = 0\}$$

and

$$N(g'(x, y)) = \{(\Delta x, \Delta y) : x\Delta x + y\Delta y = 0\}.$$

If  $(x_0, y_0)$  is in  $S$ , then directly,

$$f(x_0, y_0) = 0 \leq (x - x_0)^2 + (y - y_0)^2 = f(x, y)$$

for any  $(x, y) \in S$ , and so,  $(x_0, y_0)$  is an extreme point of  $f|_S$ . Otherwise, by Theorem 2.1 and

$$\dim N(f'(x, y)) = \dim N(g'(x, y)) = 1 \quad \forall (x, y) \in S.$$

it follows that there exists a real number  $t$  such that

$$x - x_0 = tx \quad y - y_0 = ty.$$

(Here  $t$  is just Lagrange multiplier.) So,

$$(1 - t)x = x_0, \quad \text{and} \quad (1 - t)y = y_0.$$

In addition,  $(x, y) \in S$ . We then conclude  $(1 - t)^2 = r_0^2$ . Finally, we get

$$(x, y) = \frac{1}{r_0}(x_0, y_0) \quad \text{or} \quad -\frac{1}{r_0}(x_0, y_0).$$

It is easy to examine that  $(x, y) = \frac{1}{r_0}(x_0, y_0)$  is the required extreme point.

The next example is for generalized regular constraint but not for regular.

**Example 2** Define  $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  as follows

$$g(x) = (x_1^2 + x_2^2 + x_3^2, x_3, x_3) \quad \forall x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Let  $y_0 = (1, 0, 0)$ ,  $S = g^{-1}(y_0) = \{(x_1, x_2, 0) : x_1^2 + x_2^2 = 1\}$ , and  $f(x) = (x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 + (x_3 - x_3^0)^2$  where  $(x_1^0)^2 + (x_2^0)^2 > 0$ . By Theorem 2.1 go to find the extreme point of  $f|_S$ .

It is easy to observe

$$N(f'(x)) = \{(\Delta x_1, \Delta x_2, \Delta x_3) : (x_1 - x_1^0)\Delta x_1 + (x_2 - x_2^0)\Delta x_2 - x_3^0\Delta x_3 = 0\}$$

and

$$N(g'(x)) = \{(\Delta x_1, \Delta x_2, 0) : x_1\Delta x_1 + x_2\Delta x_2 = 0\},$$

for  $x \in S$ . Obviously,  $R(g'(x)) = \{(2x_1\Delta x_1 + 2x_2\Delta x_2, \Delta x_3, \Delta x_3) : \forall (\Delta x_1, \Delta x_2, \Delta x_3) \in \mathbb{R}^3\}$  for any  $x \in S$ , so that  $\text{Rank}(g'(x)) = 2$  for all  $x \in S$ . Hence,  $S$  is the generalized regular constraint as indicated in Section 1, but not regular because of  $\text{Rank}(g'(x)) < 3$ . Now we are going to find the extreme point of  $f|_S$  by using Theorem 2.1, it follows from  $N(f'(x)) \supset N(g'(x))$

$$N_0 = \{(\Delta x_1, \Delta x_2, 0) : (x_1 - x_1^0)\Delta x_1 + (x_2 - x_2^0)\Delta x_2 = 0\} \supset N(g'(x))$$

for any  $x \in S$ . If  $(x_1^0, x_2^0, 0) \in S$  then directly,

$$f(x_1^0, x_2^0, 0) = (x_3^0)^2 \leq (x_1 - x_1^0)^2 + (x_2 - x_2^0)^3 + (x_3^0)^2 = f(x)$$

for any  $x \in S$ , and so,  $(x_1^0, x_2^0, 0)$  is an extreme point of  $f|_S$ . Otherwise,  $\dim N_0 = \dim N(g'(x)) = 1$ , so that  $N_0 = N(g'(x))$ . Thus, similar to Example 1, there exists a real number  $t$  such that

$$x_1 - x_1^0 = tx_1 \quad \text{and} \quad x_2 - x_2^0 = tx_2.$$

Let  $r_0 = \sqrt{(x_1^0)^2 + (x_2^0)^2}$ , we can get

$$x_1 = \frac{1}{r_0}x_1^0 \quad \text{and} \quad x_2 = \frac{1}{r_0}x_2^0.$$

We now conclude that  $(\frac{1}{r_0}x_1^0, \frac{1}{r_0}x_2^0, 0)$  is an extreme point of  $f|_S$ .

**Example 3** Let  $g$  and  $f$  be as the same as Example 2. Apply Theorem 2.2 to find the extreme point of  $f|_S$ .

As indicated in Example 2,

$$N(f'(x)) = \{(\Delta x_1, \Delta x_2, \Delta x_3) : (x_1 - x_1^0)\Delta x_1 + (x_2 - x_2^0)\Delta x_2 - x_3^0\Delta x_3 = 0\}$$

and

$$N(g'(x)) = \{(\Delta x_1, \Delta x_2, 0) : x_1\Delta x_1 + x_2\Delta x_2 = 0\},$$

for  $x \in S$ . Obviously,  $e_*(x) = (x_1 - x_1^0)e_1 + (x_2 - x_2^0)e_2 - x_3^0e_3$ , where  $\{e_i\}_{i=1}^3$  is an orthonormal basis in  $\mathbb{R}^3$ . By Theorem 2.2 one concludes that if  $x$  is a critical point of  $f'|_S$ , then

$$(x_1 - x_1^0)\Delta x_1 + (x_2 - x_2^0)\Delta x_2 = -x_1^0\Delta x_1 - x_2^0\Delta x_2 = 0$$

for all  $(\Delta x_1, \Delta x_2, 0) \in N(g'(x))$ . Hence  $x_1 = \lambda x_1^0$  and  $x_2 = \lambda x_2^0$  for some real number  $\lambda$ . Finally, It follows from  $(x_1, x_2, 0) \in S$  that  $x_1 = \frac{1}{r_0}x_1^0$  and  $x_2 = \frac{1}{r_0}x_2^0$  where  $r_0 = \sqrt{(x_1^0)^2 + (x_2^0)^2}$ ,

These results of Examples 1 and 2 are very intuitive. The next example is not for finding the critical point, but for showing the significance of Theorem 2.1.

**Example 4** *The constraint  $S$  and the non linear function  $f(x, y)$  are defined by*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ (both of } a \text{ and } b > 0) \quad \text{and} \quad (x - x_0)^2 + (y - y_0)^2,$$

*respectively. Let the parameter equation of  $S$  be as follows*

$$x = a \cos\theta \quad \text{and} \quad y = b \sin\theta, \quad 0 \leq \theta < 2\pi,$$

*It is known well that if  $(a \cos\theta, b \sin\theta)$  is an extreme point of  $f|_S$ , then*

$$0 = \frac{df(a \cos\theta, b \sin\theta)}{d\theta} = (b^2 - a^2) \sin\theta \cos\theta + x_0 a \sin\theta - y_0 b \cos\theta. \quad (2.2)$$

*Let us observe what yields from Theorem 2.1.*

If  $(x_0, y_0) \in S$ , it is clear

$$f(x_0, y_0) = 0 \leq (x - x_0)^2 + (y - y_0)^2 = f(x, y) \quad \forall (x, y) \in S,$$

i.e.,  $(x_0, y_0)$  is an extreme point of  $f|_S$ . So in what follows, we assume that  $(x_0, y_0)$  is not in  $S$ . Let  $g(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ . It is easy to examine that 1 is regular value of  $g$ . By direct computing,

$$N(g'(x, y)) = \{(\Delta x, \Delta y) : \frac{x}{a^2} \Delta x + \frac{y}{b^2} \Delta y = 0\}$$

and

$$N(f'(x, y)) = \{(\Delta x, \Delta y) : (x - x_0) \Delta x + (y - y_0) \Delta y = 0\}.$$

Since  $(x_0, y_0)$  is not in  $S$  we conclude  $\dim N(g'(x, y)) = \dim N(f'(x, y)) = 1$ . Thus,  $N(g'(x, y)) = N(f'(x, y))$  by Theorem 2.1. Hereby we infer

$$\frac{(x - x_0)y}{b^2} - \frac{(y - y_0)x}{a^2} = 0.$$

Replace  $x$  and  $y$  in the equation above by  $a \cos\theta$  and  $b \sin\theta$ , respectively, then we get Equation (2.2).

The example shows that Theorem 2.1 implies the classical principle (2.2).

**Remark 2.1** *In case  $F = \mathbb{R}$ , the classical principle for seeking the extreme point of  $f|_S$  is to solve Euler equations with Lagrange multiplier  $L$ ,  $f'(x) - Lg'(x) = 0$  and  $g(x) = y_0$ . In this case,  $L$  is a real number. When  $\dim F > 1$ ,  $L$  is a bounded linear functional on  $F$ . In the sequel, it will be proved that in the case of regular constraint  $S$ ,  $L$  is unique; otherwise  $L$  is ill-posed. Hence under the generalized regular but not regular constraint, it is very difficult to solve Euler equations with Lagrange multiplier. So, recently, researching of critical points under the non regular constraint has been a concerned focus in optimization theory. Therefore, Theorems 2.1 and 2.2 seem to be a new and available way.*

### 3 Lagrange Multiplier

Let  $A \in B(E, F)$  be double split. As indicated in Section 1, there exists a generalized inverse  $A^+ \in B(F, E)$  of  $A$  such that  $A^+AA^+ = A^+$  and  $AA^+A = A$ . It is easy to observe  $A^+A = P_{R(A^+)}^{N(A)}$  and  $AA^+ = P_{R(A)}^{N(A^+)}$ , where  $P_{R(A^+)}^{N(A)}$  is a projection from  $E$  onto  $R(A^+)$  and  $P_{R(A)}^{N(A^+)}$  from  $F$  onto  $R(A)$  coordinate to the following decompositions :  $E = N(A) \oplus R(A^+)$  and  $F = R(A) \oplus N(A^+)$ , respectively. So,  $I_E - P_{R(A^+)}^{N(A)} = P_{N(A)}^{R(A^+)}$  and  $I_F - P_{R(A)}^{N(A^+)} = P_{N(A^+)}^{R(A)}$ . In this section, we discuss the classical critical point principle with Lagrange multiplier  $L$  under generalized regular constraint, give an express of  $L$  by using generalized inverse and show that  $L$  is unique under regular constraint.

**Theorem 3.1** *Suppose that  $g : U \subset E \rightarrow F$  is a  $c^1$  map from open set  $U$  in Banach space  $E$  into Banach space  $F$ . Let  $g'^+(x)$  be a generalized inverse of  $g'(x)$ ,  $f$  a  $c^1$  non linear functional on  $U$ , and  $S =$  the preimage  $g^{-1}(y_0)$ ,  $y_0 \in F$ . If  $y_0$  is a generalized regular value of  $g$ , and  $x \in U$  is a critical point of  $f|_S$ , then*

$$f'(x) - f'(x) \circ g'^+(x) \circ g'(x) = 0 \quad \text{and} \quad g(x) = y_0,$$

*for any  $g'^+(x) \in GI(g'(x))$ . i.e.,  $f'(x) \circ g'^+(x) \in F^*$  is a Lagrange multiplier.*

*Proof.* Since

$$R(I_E - g'^+(x)g'(x)) = R(P_{N(g'(x))}^{R(g'^+(x))}) = N(g'(x))$$

and by Theorem 2.1,

$$0 = f'(x)(I_E - g'^+(x)g'(x)) = f'(x) - f'(x) \circ g'^+(x) \circ g'(x)$$

whenever  $x$  is a critical point of  $f|_S$ . This proves the theorem.  $\square$

In what follows, we are going to show that Lagrange multiplier is unique under regular constraint.

**Theorem 3.2** *If  $S =$  the preimage  $g^{-1}(y_0)$  is the regular constraint, i.e.,  $y_0 \in F$  is regular value of  $g$ , then the Lagrange multiplier  $L$  is unique.*

*Proof.* Assume  $x \in S$  is a critical point of  $f|_S$ . Let  $q_1(x)$  and  $q_2$  be arbitrary two generalized inverses of  $g'(x)$ . Since  $y_0$  is a regular value,  $R(g'(x)) = F$ . Hence.

$$(f'(x) \circ q_i(x))(y) = (f'(x) \circ q_i(x) \circ g'(x))(e) = f'(x)e, \quad i = 1, 2, \quad \forall y \in F,$$

where  $y = g'(x)(e)$ . Therefore  $f'(x) \circ q_1(x) = f'(x) \circ q_2(x)$ . Let  $L$  be the Lagrange multiplier. Similarly,

$$L(y) = (L \circ g'(x))(e) = f'(x)(e) = (f'(x) \circ g'^+(x))(y) \quad \forall y \in F.$$

The theorem is proved.  $\square$



## 4 Differential Construction of $GI(A)$ and Ill-Posed Lagrange Multipliers

In this section, we are going to discuss the differential construction of  $GI(A)$  and prove that  $GI(A)$  is smooth diffeomorphic to some Banach space. Then we show that when  $S$  is a generalized regular but not regular constraint, the Lagrange multiplier  $L$  is ill-posed.

Let  $A \in B(E, F)$  be double split, and  $A_0^+ \in GI(A)$  coordinating to the decompositions :  $E = N(A) \oplus R_0^+$  and  $F = R(A) \oplus N_0^+$ , i.e.,  $R_0^+ = R(A_0^+)$  and  $N_0^+ = R(I_F - AA_0^+)$ . Consider the following map  $M : B(R_0^+, N(A)) \times B(N_0^+, R(A)) \rightarrow B(F, E)$ , .

$$M(\alpha, \beta) = (I_E + \alpha)A_0^+(I_F - \beta P_{N_0^+}^{R(A)}) \quad \forall (\alpha, \beta) \in B(R_0^+, N(A)) \times B(N_0^+, R(A)).$$

**Lemma 4.1** *Let  $B = M(\alpha, \beta) \quad \forall (\alpha, \beta) \in B(R_0^+, N(A)) \times B(N_0^+, R(A))$ . The following preperities of  $M$  for any  $(\alpha, \beta) \in B(R_0^+, N(A)) \times B(N_0^+, R(A))$  hold:*

$$\begin{aligned} R(B) &= \{e + \alpha(e) : \forall e \in R_0^+\}, \\ N(B) &= \{d + \beta(d) : \forall d \in N_0^+\}, \end{aligned}$$

and  $M(\alpha, \beta) \in GI(A)$ .

*Proof.* Evidently,

$$BA = (I_E + \alpha)A_0^+(I_F - \beta P_{N_0^+}^{R(A)})A = (I_E + \alpha)A_0^+A, \text{ and } AB = AA_0^+(I_F - \beta P_{N_0^+}^{R(A)})$$

Hereby,  $ABA = A(I_E + \alpha)A_0^+A = AA_0^+A = A$ , and  $BAB = (I_E + \alpha)A_0^+AA_0^+(I_F - \beta P_{N_0^+}^{R(A)}) = B$ . This says  $B \in GI(A)$ . Note that  $e + \alpha(e) = 0$  for  $e \in R_0^+$  implies  $e = 0$ . Then we have

$$\begin{aligned} y \in N(B) &\Leftrightarrow A_0^+(I_F - \beta P_{N_0^+}^{R(A)})y = 0 \Leftrightarrow (I_F - \beta P_{N_0^+}^{R(A)})y \in N_0^+ \\ &\Leftrightarrow P_{N_0^+}^{R(A)}(I_F - \beta P_{N_0^+}^{R(A)})y = (I_F - \beta P_{N_0^+}^{R(A)})y \Leftrightarrow P_{N_0^+}^{R(A)}y = (I_F - \beta P_{N_0^+}^{R(A)})y. \end{aligned}$$

Hence  $y = P_{N_0^+}^{R(A)}y + \beta(P_{N_0^+}^{R(A)}y)$  i.e.,  $N(B) \subset \{d + \beta(d) : \forall d \in N_0^+\}$ . Conversely, let  $y = d + \beta(d)$  for any  $d \in N_0^+$ , then  $P_{N_0^+}^{R(A)}y = d$  and so  $y = P_{N_0^+}^{R(A)}y + \beta(P_{N_0^+}^{R(A)}y)$ . Thus, by the equivalent relations above,  $y \in N(B)$ . This shows  $N(B) = \{d + \beta(d) : \forall d \in N_0^+\}$ . Let  $x \in R(B)$ ,  $x = (I_E + \alpha)A_0^+(I_F - \beta P_{N_0^+}^{R(A)})y$ , and  $e = A_0^+(I_F - \beta P_{N_0^+}^{R(A)})y$ . Then  $x = e + \alpha(e)$  and so, we prove  $R(B) \subset \{e + \alpha(e) : \forall e \in R_0^+\}$ . Conversely, let  $x = e + \alpha(e)$  where  $e \in R_0^+$ , and  $e = A_0^+y$ ,  $y \in R(A)$ . Then

$$x = (I_E + \alpha)A_0^+y = (I_E + \alpha)A_0^+(I_F - \beta P_{N_0^+}^{R(A)})y$$

since  $P_{N_0^+}^{R(A)}y = 0$ . Now, one concludes  $R(B) = \{e + \alpha(e) : \forall e \in R_0^+\}$ .  $\square$

**Lemma 4.2**  $M(\alpha, \beta) : B(R_0^+, N(A)) \times B(N_0^+, R(A)) \rightarrow GI(A)$  is bijective.

*Proof.* It is easy to see that if  $B, B_1 \in GI(A)$ ,  $R(B) = R(B_1)$  and  $N(B) = N(B_1)$ , then  $B = B_1$ . In fact,

$$\begin{aligned} B &= BAB = BP_{R(A)}^{N(B)} = BP_{R(A)}^{N(B_1)} = BAB_1 \\ &= P_{R(B)}^{N(A)} B_1 = P_{R(B_1)}^{N(A)} B_1 = B_1 AB_1 = B_1. \end{aligned}$$

Hereby, due to the conclusions about  $R(B)$  and  $N(B)$  in Lemma 4.1 one can infer that  $M(\alpha, \beta)$  is injective. Let both of  $B$  and  $A_0^+$  be in  $GI(A)$ . we have

$$P_{R_0^+}^{N(A)} P_{R(B)}^{N(A)} e = P_{R_0^+}^{N(A)} (P_{R(B)}^{N(A)} e + P_{N(A)}^{R(B)} e) = P_{R_0^+}^{N(A)} e = e \quad \forall e \in R_0^+$$

and

$$P_{R(B)}^{N(A)} P_{R_0^+}^{N(A)} e = P_{R(B)}^{N(A)} (P_{R_0^+}^{N(A)} e + P_{N(A)}^{R_0^+} e) = P_{R(B)}^{N(A)} e = e \quad \forall e \in R(B).$$

Similarly,

$$P_{N_0^+}^{R(A)} P_{N(B)}^{R(A)} d = d \quad \forall d \in N_0^+ \quad \text{and} \quad P_{N(B)}^{R(A)} P_{N_0^+}^{R(A)} d = d \quad \forall d \in N(B).$$

Let

$$\alpha = P_{N(A)}^{R_0^+} P_{R(B)}^{N(A)} \quad \text{and} \quad \beta = P_{R(A)}^{N_0^+}$$

for any  $B \in GI(A)$ . It is easy to check

$$R(B) = \{e + \alpha(e) : \forall e \in R_0^+\} \quad \text{and} \quad N(B) = \{d + \beta(d) : \forall d \in N_0^+\}.$$

In fact,

$$h = P_{R_0^+}^{N(A)} h + P_{N(A)}^{R_0^+} h = P_{R_0^+}^{N(A)} h + P_{N(A)}^{R_0^+} P_{R(B)}^{N(A)} P_{R_0^+}^{N(A)} h = P_{R_0^+}^{N(A)} h + \alpha(P_{R_0^+}^{N(A)} h)$$

for any  $h \in R(B)$ , so that  $R(B) \subset \{e + \alpha(e) : \forall e \in R_0^+\}$ ; conversely,

$$h = e + \alpha(e) = P_{R_0^+}^{N(A)} P_{R(B)}^{N(A)} e + P_{N(A)}^{R_0^+} P_{R(B)}^{N(A)} e = P_{R(B)}^{N(A)} e \in R(B)$$

for any  $e \in R_0^+$ . Similarly, one can verify the above relation about  $N(B)$ . By Lemma 4.1 it follows  $B = M(\alpha, \beta)$  since both of  $B$  and  $M(\alpha, \beta)$  are in  $GI(A)$ . Therefore,  $M : B(R_0^+, N(A)) \times B(N_0^+, R(A)) \rightarrow GI(A)$  is bijective.  $\square$

**Theorem 4.1** The mapping  $M(\alpha, \beta)$  from  $B(R_0^+, N(A)) \times B(N_0^+, R(A))$  onto  $GI(A)$  is smooth diffeomorphism.

*Proof.* By direct computing

$$\begin{aligned}
& M(\alpha + \Delta\alpha, \beta + \Delta\beta) \\
&= (I_E + \alpha)A_0^+(I_F - (\beta + \Delta\beta))P_{N_0^+}^{R(A)} + \Delta\alpha A_0^+(I_F - (\beta + \Delta\beta))P_{N_0^+}^{R(A)} \\
&= M(\alpha, \beta) - (I_E + \alpha)A_0^+\Delta\beta P_{N_0^+}^{R(A)} + \Delta\alpha A_0^+(I_F - \beta P_{N_0^+}^{R(A)}) - \Delta\alpha A_0^+\Delta\beta P_{N_0^+}^{R(A)}.
\end{aligned}$$

Hereby we infer

$$\begin{aligned}
& DM(\alpha, \beta) < \Delta\alpha, \Delta\beta > \\
&= \Delta\alpha A_0^+(I_F - \beta P_{N_0^+}^{R(A)}) - (I_E + \alpha)A_0^+\Delta\beta P_{N_0^+}^{R(A)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& DM(\alpha + \Delta\alpha_1, \beta + \Delta\beta_1) < \Delta\alpha, \Delta\beta >, \\
&= \Delta\alpha A_0^+(I_F - (\beta + \Delta\beta_1))P_{N_0^+}^{R(A)} - (I_E + (\alpha + \Delta\alpha_1))A_0^+\Delta\beta P_{N_0^+}^{R(A)} \\
&= \Delta\alpha A_0^+(I_F - \beta)P_{N_0^+}^{R(A)} - \Delta\alpha A_0^+\Delta\beta_1 P_{N_0^+}^{R(A)} - (I_E + \alpha)A_0^+\Delta\beta P_{N_0^+}^{R(A)} - \Delta\alpha_1 A_0^+\Delta\beta P_{N_0^+}^{R(A)} \\
&= DM(\alpha, \beta) < \Delta\alpha, \Delta\beta > - \Delta\alpha A_0^+\Delta\beta_1 P_{N_0^+}^{R(A)} - \Delta\alpha_1 A_0^+\Delta\beta P_{N_0^+}^{R(A)}.
\end{aligned}$$

Hereby we conclude

$$\begin{aligned}
& D^2M(\alpha, \beta) < (\Delta\alpha, \Delta\beta), (\Delta\alpha_1, \Delta\beta_1) > \\
&= -\Delta\alpha A_0^+\Delta\beta_1 P_{N_0^+}^{R(A)} - \Delta\alpha_1 A_0^+\Delta\beta P_{N_0^+}^{R(A)}.
\end{aligned}$$

Note that  $D^2M(\alpha, \beta)$  is independent of  $\alpha$  and  $\beta$ . Hence

$$D^n(\alpha, \beta) = 0 \quad n \geq 3.$$

The theorem is proved due to Lemma 4.2.  $\square$

For the details see [M4] and [M5].

We are now in the position to consider the ill-posed Lagrange multiplier in Euler equations under non regular constraint.

**Theorem 4.2** *Suppose that  $g : U \subset E \rightarrow F$  is a  $c^1$  map, and  $S =$  the preimage  $g^{-1}(y_0)$  is a generalized regular constraint but not regular, i.e.,  $y_0 \in F$  is a generalized regular value of  $g$  but not regular. Let  $f$  be a non linear functional defined on  $U$ . Then the Lagrange multiplier  $L$  in Euler equations is ill-posed.*

*Proof.* Let  $x$  be a critical point of  $f|_S$ , and  $e_0 \in E$  such that  $f'(x)e_0 = 1$ . Since  $x$  is a critical point of  $f|_S$  by Theorem 3.1,

$$1 = (f'(x) \circ g'^+(x) \circ g'(x))(e_0) \quad \text{for any fixed } g'^+(x) \in GI(g'(x)).$$

Let  $y_1 = g'(x)e_0 \in R(g'(x))$ . Since  $x \in S$  and  $S$  is the non regular constraint, there exists a non zero  $y^+$  in  $N(g'^+(x))$ . Write  $N(g'^+(x))$  by  $N_0^+$ , and let  $N_0^+ = [y^+] \oplus N^+$ , where  $[y^+]$  denotes the subspace generated by  $y^+$ . Define  $\beta \in B(N_0^+, R(g'(x)))$  as follows

$$\beta(y) = 0 \quad \text{if} \quad y \in N^+; \quad \text{else} \quad \beta(y^+) = y_1.$$

By Lemma 4.1,

$$B = g'(x)(I_F - \beta P_{N_0^+}^{R(g'(x))}) \in GI(g'(x)) \quad \text{and} \quad y^+ + y_1 \in N(B).$$

Now we see that two deferent Lagrange multipliers at the critical point  $x$ ,  $L = f'(x) \circ g'^+(x)$  and  $L_1 = f'(x) \circ B$ , both are in  $F^*$  and satisfy

$$L = (f'(x) \circ g'^+(x))(y^+ + y_1) = (f'(x) \circ g'^+(x))(y_1) = ((f'(x) \circ g'^+(x))(g'(x))e_0) = 1,$$

and

$$L_1 = (f'(x) \circ B)((y^+ + y_1) = 0.$$

The theorem is proved.  $\square$

The constraint  $S$  in Example 2 is non regular. Now take it with  $x_3^0 \neq 0$  for example to illustrate the ill-posed Lagrange multiplier in case of non regular constraint.

Let  $\{e_i\}_1^3$  and  $\{\varepsilon_i\}_1^3$  denote orthonormal bases of  $\mathbb{R}^3$  containing the domain and the range of  $g$ , respectively. As is indicated in Example 2 that  $x_0 = \frac{1}{r_0}(x_1^0, x_2^0, 0)$  is a critical point of  $f|_S$ . By direct computing,

$$g'(x_0)(x_1e_1 + x_2e_2 + x_3e_3) = \frac{1}{r_0}(x_1^0x_1 + x_2^0x_2)\varepsilon_1 + x_3\varepsilon_2 + x_3\varepsilon_3$$

and

$$f'(x_0)(y_1\varepsilon_1 + y_2\varepsilon_2 + y_3\varepsilon_3) = 2\left(\frac{2}{r_0} - 1\right)y_1 + 2\left(\frac{2}{r_0} - 1\right)y_2 - 2x_3^0y_3.$$

Hereby,

$$N(g'(x_0)) = \{x_1e_1 + x_2e_2 : x_1^0x_1 + x_2^0x_2 = 0\},$$

and so,

$$N(g'(x_0)) \perp (x_1^0e_1 + x_2^0e_2) \perp x_3^0e_3.$$

Then, we have the following decomposition of  $\mathbb{R}^3$  containing the domain of  $g$

$$\mathbb{R}^3 = N(g'(x_0)) \oplus [x_1^0e_1 + x_2^0e_2] \oplus [e_3],$$

where  $[.]$  denotes the subspace generated by the vector in the bracket, and  $\oplus$  orthogonal sum. Moreover, since  $g'(x_0)(x_1^0e_1 + x_2^0e_2) = 2r_0\varepsilon_1$  and  $g'(x_0)(x_3^0e_3) = x_3^0\varepsilon_2 + x_3^0\varepsilon_3$  it follows

$$R(g'(x_0)) = [2r_0\varepsilon_1] \oplus [x_3^0\varepsilon_2 + x_3^0\varepsilon_3].$$

Hereby we infer the decomposition of  $\mathbb{R}^3$  containing the range of  $g$ :

$$\mathbb{R}^3 = R(g'(x_0)) \oplus [-x_3^0 \varepsilon_2 + x_3^0 \varepsilon_3].$$

Let  $N_0^+ = [-x_3^0 \varepsilon_2 + x_3^0 \varepsilon_3]$ . Now we can define  $g'(x_0)^+ \in GI(g'(x_0))$  as follows,

$$\begin{aligned} g'(x_0)^+ y &= 0 \quad \text{if } y \in N_0^+, \\ &= x_1^0 e_1 + x_2^0 e_2 \quad \text{if } y = 2x_0^0 \varepsilon_1, \\ &= x_3^0 e_3 \quad \text{if } y = x_3^0 \varepsilon_2 + x_3^0 \varepsilon_3. \end{aligned}$$

Obviously,  $L = f'(x_0) \circ g'(x_0)^+ \in F^*$  and

$$\begin{aligned} L(2x_3^0 \varepsilon_3) &= (f'(x_0) \circ g'(x_0)^+)((-x_3^0 \varepsilon_2 + x_3^0 \varepsilon_3) + (x_3^0 \varepsilon_2 + x_3^0 \varepsilon_3)) \\ &= (f'(x_0) \circ g'(x_0)^+)(x_3^0 \varepsilon_2 + x_3^0 \varepsilon_3) \\ &= f'(x_0)(x_3^0 e_3) = -2(x_3^0)^2 \neq 0. \end{aligned}$$

Let  $\beta \in B(N_0^+, R(g'(x_0)))$  such that

$$\beta(-x_3^0 \varepsilon_2 + x_3^0 \varepsilon_3) = x_3^0 \varepsilon_2 + x_3^0 \varepsilon_3.$$

Set

$$B = g'(x_0)^+(I_F - \beta P_{N_0^+}^{R(g'(x_0))}).$$

By Lemma 4.1,

$$2x_3^0 \varepsilon_3 = (-x_3^0 \varepsilon_2 + x_3^0 \varepsilon_3) + (x_3^0 \varepsilon_2 + x_3^0 \varepsilon_3) \in N(B).$$

(This conclusion can also be to check directly.) Hence

$$L_1(2x_3^0 \varepsilon_3) = (f'(x_0) \circ B)(2x_3^0 \varepsilon_3) = 0.$$

Therefore  $L$  and  $L_1$  are two deferent Lagrange multipliers.

**Remark 4.1** *It is not enough for the generalized preimage theorem to express several constraints in optimization theory, finance mathematics and so on. A generalization to Thom's famous result for transversality, generalized transversality theorem is helpful, (see [M2]). We can also have similar principle to Theorem 2.1, which will be discussed else where.*

## References

[AMR] Abraham R, Marsden J. E, Ratiu T. *Manifold, Tensor analysis and its Applications*, WPC: Springer-Verlag, (1988).

- [B] Berger M, *Nonlinearity and Functional Analysis*, New York Academic Press, (1976).
- [HM] Qianlian H, Jipu Ma, *Perturbation Analysis of Generalized Inverses of Linear Operators in Banach Space*, Linear Algebra Appl. 389 (2004),335-364.
- [M1] Jipu Ma , *Three classes of smooth Banach manifolds in  $B(E,F)$* , Sci.China Ser.A 50:9(2007), 1233-1239.
- [M2] Jipu Ma , *A generalized transversality in global analysis*, PJM 236:2(2008), 357-371.
- [M3] Jipu Ma , *Complete rank theorem of advanced calculus and singularities of bounded linear operators*, Front.Math. china 3:2 (2008), 304-316.
- [M4] Zhaofeng Ma ,Jipu Ma , *The Smooth Banach Submanifold  $B^*(E, F)$  in  $B(E, F)$* , Sci.China Ser.A 52:11(2009), 2479-2492.
- [M5] Zhaofeng Ma , Jipu Ma , *A Common Property of  $R(E, F)$  and  $\mathbb{R}^n, \mathbb{R}^m$  and A New Method for Seeking A Path to Connect Two Operators*, Sci.China Ser.A 53:10(2010), 2605-2620.
- [M6] Jipu Ma , *A generalized preimage theorem in global nalysis*, Science in China, Ser.A, 44:(2001), 299-303.
- [M7] Jipu Ma , *(1,2)-Inverse of Operators between Banach Spaces and Local Conjugacy Theorem*, Chi. Ann. Math.. 20(B):1 (1999), 57-62.
- [M8] Jipu Ma , *Local Conjugacy Theorem, Rank Theorems in Advanced Calculus and A Generalized Principle for Constructing Banagh Manifold* , Science in China, Ser.A, 43:12 (2000), 1233-1237.
- [M9] Jipu Ma, *Rank Theorem of Operators Between Banach Spaces* , Science in China, Ser.A, 43:1(2000), 1-5.
- [Z] Zeidler E, *Nonlinear Functional Analysis and Its Applications IV*, New York-Berlin: Springer-Verlag, (1988).
- [N] Nashed M. Z, *Generalized inverses and applications*, Academic Press, New York, San-Francisco, London 1976.

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